Large Sample Estimators of the Stochastic Discount Factor

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Outline

Goal

1. Improve performance of an "agnostic" SDF estimator by imposing additional structure on nature of asset returns.
2. Adjusting estimators for small sample biases.
3. Compare estimators by simulating economies and comparing the relation between estimators and the true SDF.

Outline

Agnostic SDF Estimator

Lack of arbitrage implies that there is a positive stochastic discount factor, m_t, in the space of portfolio returns, such that

$$\mathbb{E}[m_t R^g_{i,t}] = 1 \tag{1}$$

where $R^g_{i,t}$ be the gross return on asset i in period t. Rational expectations imply

$$m_t R^g_{i,t} = \mathbb{E}[m_t R^g_{i,t}] + \varepsilon_{i,t} = 1 + \varepsilon_{i,t} \tag{2}$$

where $\mathbb{E}[\varepsilon_{i,t}] = 0$.

Agnostic SDF Estimator

Given a time series $t = 1, \ldots T$:

$$\frac{1}{T}\sum_{t=1}^{T} m_t R_{i,t}^{g} = 1 + \frac{1}{T}\sum_{t=1}^{T} \varepsilon_{i,t} = 1 + \bar{\varepsilon}_i \to 1 \tag{3}$$

as $T \to \infty$.

Agnostic SDF Estimator

Let $\mathbf{R}^g$ be the $N \times T$ matrix of gross asset returns with rank T, m be the $T \times 1$ vector of time series observations on the stochastic discount factor, $\bar{\varepsilon}$ be the $N \times 1$ vector of average pricing errors, and $\mathbf{1}_N$ be an $N \times 1$ vector of ones. Equations (2) and (3) imply that

$$\mathbf{R}^g \mathbf{m}/T = \mathbf{1}_N + \bar{\varepsilon} \approx \mathbf{1}_N$$

The Pukthuanthong and Roll (2016) (PR-SDF) "agnostic" estimator $\mathbf{m}^{\mathrm{PR}}$ is the vector that minimizes the sum of squared pricing errors, $\bar{\varepsilon}'\bar{\varepsilon}$ and is given by:

$$\mathbf{m}^{\mathrm{PR}} = T\left(\mathbf{R}^{g\prime}\mathbf{R}^g\right)^{-1}\mathbf{R}^{g\prime}\mathbf{1}_N = \left(\frac{\mathbf{R}^{g\prime}\mathbf{R}^g}{T}\right)^{-1}\left(\mathbf{R}^{g\prime}\mathbf{1}_N\right) \tag{4}$$

Agnostic SDF Estimator

Since $\mathbf{1}_N = \mathbf{R}^g \mathbf{m} - \bar{\varepsilon}$

$$\mathbf{m}^{\mathrm{PR}} = \mathbf{m} - \left(\frac{\mathbf{R}^{g\prime} \mathbf{R}^g}{T} \right)^{-1} (\mathbf{R}^{g\prime} \bar{\varepsilon}) \tag{5}$$

Issues

1. Lack of structure imposed on SDF induces a large amount of estimation error

2. the PR-SDF is biased in small time-series samples since

$$\mathbb{E}[\mathbf{R}^{g}\bar{\varepsilon}] \neq 0$$

Outline

Imposing a Factor Structure on Gross Returns

1. The term $\left(\frac{\mathbf{R}^{g\prime}\mathbf{R}^{g}}{T}\right)$ in (4) is proportional to the matrix whose eigenvectors converge to systematic factors in the Asymptotic Principal Components procedure of Connor and Korajczyk (1986).
2. The PR-SDF is equivalent to using T factors (e.g., 120 factors if we have 10 years of monthly data).
3. We propose using $K << T$ factors.

Imposing a Factor Structure on Gross Returns

Define the $T \times K$ matrix of $\mathbf{P}_K^g$ such that the k-th column of $\mathbf{P}_K^g$ is the eigenvector corresponding to the k-th largest eigenvalue of $\frac{\mathbf{R}^{g\prime}\mathbf{R}^g}{N}$.
Our factor based SDF estimator is proportional to $\mathbf{P}_K^g$

$$\mathbf{m}_K^g = \mathbf{P}_K^g \boldsymbol{\delta}^g, \tag{6}$$

with the $(K \times 1)$ vector $\boldsymbol{\delta}^g$ chosen to minimize the pricing error

$$\frac{\mathbf{R}^g \mathbf{P}_K^g}{T} \boldsymbol{\delta}^g - \mathbf{1}_N$$

Imposing a Factor Structure on Gross Returns

Which yields

$$\boldsymbol{\delta}^g = \left(\left(\frac{\mathbf{R}^g \mathbf{P}_K^g}{T} \right)' \left(\frac{\mathbf{R}^g \mathbf{P}_K^g}{T} \right) \right)^{-1} \left(\left(\frac{\mathbf{R}^g \mathbf{P}_K^g}{T} \right)' \mathbf{1}_N \right). \tag{7}$$

So

$$\mathbf{m}_K^g = \mathbf{P}_K^g \left(\left(\frac{\mathbf{R}^g \mathbf{P}_K^g}{T} \right)' \left(\frac{\mathbf{R}^g \mathbf{P}_K^g}{T} \right) \right)^{-1} \left(\left(\frac{\mathbf{R}^g \mathbf{P}_K^g}{T} \right)' \mathbf{1}_N \right). \tag{8}$$

Imposing a Factor Structure on Excess Returns

Let $\mathbf{R}^e$ be the $N \times T$ matrix of excess returns, $\mathbf{P}^e_K$ is the matrix of the K largest eigenvalue of $\frac{\mathbf{R}^{e\prime}\mathbf{R}^e}{N}$, and $\mathbf{R}_f^{-1}$ be the $T \times 1$ vector of which the t-th element is $R_{f,t}^{-1}$, the inverse of the gross riskless rate of return.

$$\mathbf{m}^e_K = \mathbf{R}_f^{-1} + \mathcal{P}\mathbf{P}^e_K \boldsymbol{\delta}^e \tag{9}$$

$$\boldsymbol{\delta}^e = -\left(\left(\frac{\mathbf{R}^e\mathcal{P}\mathbf{P}^e_K}{T}\right)'\left(\frac{\mathbf{R}^e\mathcal{P}\mathbf{P}^e_K}{T}\right)\right)^{-1}\left(\left(\frac{\mathbf{R}^e\mathcal{P}\mathbf{P}^e_K}{T}\right)'\left(\frac{\mathbf{R}^e\mathbf{R}_f^{-1}}{T}\right)\right). \tag{10}$$

where $\mathcal{P} = \mathbf{I}_T - \frac{\mathbf{R}_f\mathbf{R}_f'}{\mathbf{R}_f'\mathbf{R}_f}$ is a projection matrix which is orthogonal to $\mathbf{R}_f$.

Outline

Small Sample Bias Correction

As in Connor and Korajczyk (1986), we assume that the cross-sectional average idiosyncratic variance, s, is independent of time and let $\hat{s}$ be an N-consistent estimate of s.

Our adjusted rotations of the eigenvectors are:

$$\mathbf{m}^g_{adj} = \mathbf{P}^g_K \boldsymbol{\delta}^g_{adj} \text{ and } \mathbf{m}^e_{adj} = \mathbf{R}_f^{-1} + \mathcal{P}\mathbf{P}^e_K \boldsymbol{\delta}^e_{adj},$$

$$\boldsymbol{\delta}^g_{adj} = \left(\frac{1}{N} \left(\frac{\mathbf{R}^g \mathbf{P}^g_K}{T} \right)' \left(\frac{\mathbf{R}^g \mathbf{P}^g_K}{T} \right) - \frac{\widehat{s}}{T^2} \mathbf{I}_K \right)^{-1} \left(\frac{1}{N} \left(\frac{\mathbf{R}^g \mathbf{P}^g_K}{T} \right)' \mathbf{1}_N \right),$$

$$\boldsymbol{\delta}^e_{adj} = -\left(\frac{1}{N} \left(\frac{\mathbf{R}^e \mathcal{P} \mathbf{P}^e_K}{T} \right)' \left(\frac{\mathbf{R}^e \mathcal{P} \mathbf{P}^e_K}{T} \right) - \frac{\widehat{s}}{T^2} \mathbf{P}^e_K \mathcal{P} \mathbf{P}^{e\prime}_K \right)^{-1} \left(\frac{1}{N} \left(\frac{\mathbf{R}^e \mathcal{P} \mathbf{P}^e_K}{T} \right)' \left(\frac{\mathbf{R}^e \mathbf{R}_f^{-1}}{T} \right) - \frac{\widehat{s}}{T^2} \mathbf{P}^{e\prime}_K \mathcal{P} \mathbf{R}_f^{-1} \right)$$

This is similar to the bias corrections in Litzenberger and Ramaswamy (1979) and Shanken (1992).

Outline

Simulation

1. Simulated economy with $N =$2,000; 4,000; and 8,000
2. "Monthly" asset returns with $T =$60, 120, 240, and 480 (corresponding to 5, 10, 20, and 40 years of data).
3. Returns constructed to obey one of three asset pricing models, the CAPM and the Fama-French 3- and 5-factor models.
4. Cross-sectional systematic and idiosyncratic risk exposures drawn jointly from the empirical distribution, as in Chen, Connor, and Korajczyk (2017).

Performance metrics

Regress the SDF estimator, $\hat{m}_t$, on the true SDF, m_t: $\hat{m}_t = a + bm_t + u_t$.

1. R^2 =1
2. $a = 0$
3. $b = 1$

Results from gross returns: R^2

$T\backslash N$	$\mathbf{m}_{\mathrm{PR}}$ 2000	$\mathbf{m}_{\mathrm{PR}}$ 4000	$\mathbf{m}_{\mathrm{PR}}$ 8000	$\mathbf{m}^g$ 2000	$\mathbf{m}^g$ 4000	$\mathbf{m}^g$ 8000	$\mathbf{m}^g_{adj}$ 2000	$\mathbf{m}^g_{adj}$ 4000	$\mathbf{m}^g_{adj}$ 8000
				Panel A: CAPM					
60	0.09	0.11	0.17	0.66	0.50	0.68	0.37	0.87	0.92
120	0.07	0.09	0.14	0.73	0.48	0.75	0.45	0.89	0.93
240	0.04	0.06	0.11	0.79	0.47	0.84	0.55	0.91	0.95
480	0.02	0.04	0.07	0.85	0.46	0.92	0.66	0.94	0.97
				Panel B: FF3					
60	0.23	0.30	0.45	0.56	0.47	0.73	0.15	0.50	0.54
120	0.18	0.25	0.39	0.67	0.57	0.82	0.12	0.64	0.68
240	0.12	0.19	0.31	0.77	0.69	0.89	0.09	0.77	0.80
480	0.07	0.13	0.21	0.82	0.78	0.92	0.06	0.85	0.88
				Panel C: FF5					
60	0.39	0.49	0.59	0.60	0.61	0.66	0.28	0.51	0.55
120	0.32	0.44	0.56	0.73	0.72	0.74	0.36	0.67	0.70
240	0.23	0.35	0.49	0.80	0.79	0.78	0.44	0.79	0.82
480	0.14	0.25	0.37	0.84	0.83	0.82	0.51	0.86	0.89

Results from gross returns: *intercept*

$T \backslash N$	$\mathbf{m}_{\text{PR}}$ 2000	$\mathbf{m}_{\text{PR}}$ 4000	$\mathbf{m}_{\text{PR}}$ 8000	$\mathbf{m}^g$ 2000	$\mathbf{m}^g$ 4000	$\mathbf{m}^g$ 8000	$\mathbf{m}^g_{adj}$ 2000	$\mathbf{m}^g_{adj}$ 4000	$\mathbf{m}^g_{adj}$ 8000
Panel A: CAPM									
60	0.22	0.54	0.76	0.03	0.98	0.93	1.33	0.06	0.00
120	0.17	0.40	0.49	0.29	1.05	0.67	1.38	0.05	0.02
240	0.14	0.74	0.30	0.23	2.24	0.41	1.40	0.05	0.02
480	0.12	0.16	0.15	0.21	1.24	0.20	1.41	0.03	0.02
Panel B: FF3									
60	0.50	0.49	0.48	0.62	0.65	0.55	0.77	0.09	0.03
120	0.34	0.32	0.31	0.48	0.51	0.38	0.81	0.07	0.02
240	0.20	0.19	0.18	0.34	0.36	0.24	0.85	0.05	0.03
480	0.11	0.10	0.10	0.24	0.25	0.15	0.87	0.05	0.03
Panel C: FF5									
60	0.49	0.46	0.48	0.60	0.56	0.56	0.46	0.10	0.04
120	0.31	0.29	0.31	0.44	0.40	0.42	0.45	0.07	0.02
240	0.18	0.17	0.19	0.31	0.28	0.31	0.43	0.05	0.02
480	0.10	0.09	0.10	0.22	0.20	0.24	0.41	0.05	0.02

Results from gross returns: *slope*

$T \backslash N$	$\mathbf{m}_{\mathrm{PR}}$ 2000	$\mathbf{m}_{\mathrm{PR}}$ 4000	$\mathbf{m}_{\mathrm{PR}}$ 8000	$\mathbf{m}^g$ 2000	$\mathbf{m}^g$ 4000	$\mathbf{m}^g$ 8000	$\mathbf{m}^g_{adj}$ 2000	$\mathbf{m}^g_{adj}$ 4000	$\mathbf{m}^g_{adj}$ 8000
Panel A: CAPM									
60	0.78	0.45	0.24	0.97	0.01	0.06	-0.35	0.96	1.01
120	0.83	0.59	0.50	0.70	-0.06	0.32	-0.39	0.96	0.98
240	0.86	0.25	0.70	0.77	-1.25	0.59	-0.41	0.96	0.98
480	0.88	0.84	0.85	0.78	-0.25	0.80	-0.42	0.97	0.99
Panel B: FF3									
60	0.50	0.50	0.51	0.37	0.34	0.44	0.23	0.93	0.98
120	0.66	0.67	0.68	0.52	0.49	0.61	0.19	0.94	0.98
240	0.80	0.81	0.81	0.66	0.63	0.75	0.14	0.95	0.98
480	0.89	0.90	0.90	0.76	0.75	0.84	0.12	0.96	0.98
Panel C: FF5									
60	0.50	0.53	0.51	0.39	0.43	0.43	0.54	0.91	0.98
120	0.68	0.71	0.68	0.56	0.59	0.58	0.55	0.94	0.99
240	0.82	0.83	0.81	0.69	0.72	0.68	0.57	0.95	0.98
480	0.90	0.91	0.89	0.78	0.80	0.76	0.59	0.95	0.98

Results from excess returns: R^2

	$\mathbf{m}_{\text{PR}}$			$\mathbf{m}^e$			$\mathbf{m}^e_{adj}$		
$T \backslash N$	2000	4000	8000	2000	4000	8000	2000	4000	8000
				Panel A: CAPM					
60	0.09	0.11	0.17	0.99	0.99	1.00	0.97	1.00	1.00
120	0.07	0.09	0.14	0.99	0.99	1.00	0.98	1.00	1.00
240	0.04	0.06	0.11	0.99	0.99	1.00	0.99	1.00	1.00
480	0.02	0.04	0.07	0.99	1.00	1.00	0.99	1.00	1.00
				Panel B: FF3					
60	0.23	0.30	0.45	0.78	0.68	0.86	0.31	0.54	0.56
120	0.18	0.25	0.39	0.86	0.77	0.92	0.37	0.68	0.69
240	0.12	0.19	0.31	0.92	0.85	0.95	0.42	0.80	0.81
480	0.07	0.13	0.21	0.94	0.91	0.97	0.46	0.89	0.90
				Panel C: FF5					
60	0.39	0.49	0.59	0.73	0.71	0.82	0.38	0.53	0.55
120	0.32	0.44	0.56	0.82	0.80	0.90	0.50	0.69	0.71
240	0.23	0.35	0.49	0.88	0.85	0.94	0.60	0.81	0.83
480	0.14	0.25	0.37	0.91	0.89	0.96	0.67	0.88	0.90

Results from excess returns: *intercept*

	$\mathbf{m}_{\mathrm{PR}}$			$\mathbf{m}^{e}$			$\mathbf{m}^{e}_{adj}$		
$T \setminus N$	2000	4000	8000	2000	4000	8000	2000	4000	8000
				Panel A: CAPM					
60	0.22	0.54	0.76	0.14	0.17	0.15	-0.01	-0.05	-0.05
120	0.17	0.40	0.49	0.06	0.08	0.13	0.00	-0.01	-0.01
240	0.14	0.74	0.30	0.04	-0.05	0.04	0.00	0.00	0.00
480	0.12	0.16	0.15	0.03	0.03	0.02	0.00	0.00	0.00
				Panel B: FF3					
60	0.50	0.49	0.48	0.39	0.43	0.35	0.37	-0.01	-0.02
120	0.34	0.32	0.31	0.26	0.29	0.22	0.40	0.01	-0.01
240	0.20	0.19	0.18	0.16	0.18	0.13	0.43	0.01	0.01
480	0.11	0.10	0.10	0.11	0.11	0.08	0.46	0.01	0.01
				Panel C: FF5					
60	0.49	0.46	0.48	0.47	0.45	0.41	0.23	0.02	-0.01
120	0.31	0.29	0.31	0.32	0.31	0.26	0.25	0.03	0.00
240	0.18	0.17	0.19	0.21	0.21	0.16	0.26	0.03	0.01
480	0.10	0.09	0.10	0.14	0.15	0.09	0.25	0.03	0.01

Results from excess returns: *slope*

$T \backslash N$	$\mathbf{m}_{\mathrm{PR}}$ 2000	$\mathbf{m}_{\mathrm{PR}}$ 4000	$\mathbf{m}_{\mathrm{PR}}$ 8000	$\mathbf{m}^e$ 2000	$\mathbf{m}^e$ 4000	$\mathbf{m}^e$ 8000	$\mathbf{m}^e_{adj}$ 2000	$\mathbf{m}^e_{adj}$ 4000	$\mathbf{m}^e_{adj}$ 8000
				Panel A: CAPM					
60	0.78	0.45	0.24	0.86	0.83	0.85	1.03	1.07	1.07
120	0.83	0.59	0.50	0.94	0.92	0.87	1.01	1.02	1.02
240	0.86	0.25	0.70	0.96	1.05	0.96	1.00	1.00	1.00
480	0.88	0.84	0.85	0.97	0.97	0.98	1.00	1.00	1.00
				Panel B: FF3					
60	0.50	0.50	0.51	0.60	0.57	0.64	0.64	1.03	1.04
120	0.66	0.67	0.68	0.74	0.71	0.78	0.61	1.00	1.01
240	0.80	0.81	0.81	0.84	0.82	0.87	0.57	0.99	1.00
480	0.89	0.90	0.90	0.89	0.89	0.92	0.54	0.99	0.99
				Panel C: FF5					
60	0.50	0.53	0.51	0.52	0.55	0.58	0.78	0.99	1.03
120	0.68	0.71	0.68	0.68	0.69	0.74	0.75	0.98	1.01
240	0.82	0.83	0.81	0.79	0.79	0.84	0.75	0.98	0.99
480	0.90	0.91	0.89	0.86	0.85	0.91	0.75	0.97	0.99

Outline

Conclusion

We propose several extensions to the Pukthuanthong and Roll (2016) "agnostic" SDF estimator that:

- Impose more structure on the SDF by imposing a factor structure.
- Provide a bias correction for SDF estimator induced by small time series samples though noise in estimating mean returns.

We simulate asset returns in economies with exact factor pricing and compare the alternative SDF estimators to the true SDF.

Our proposed alternatives perform well relative to the "agnostic" estimator when using sample sizes that are as large, or possibly larger, than those used in most empirical work.

Extensions

All of these estimators assume balanced panels. We have developed versions that allow for varying cross-sectional sample across time.

- We need to use the small-sample bias correction for this.

We plan on applying the estimators to time portfolios with varying composition based on instruments that predict the cross section of returns.

We will apply the alternate SDF estimators to actual returns data to compare their ability to explain the returns on various anomaly portfolios