Optimal Window Selection for Forecasting in The Presence of Recent Structural Breaks

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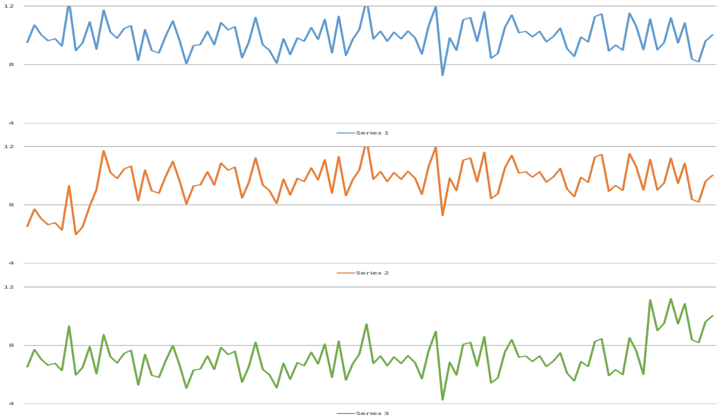
Motivation

- The presence of structure breaks is a crucial issue in forecasting
  - including pre-break data may lead to biased parameter estimates and biased forecasts
  - however reducing sample size increases the variance of the parameter estimates, which maps into the forecast errors
- Trade-off between the bias and variance $\rightleftharpoons$ Optimal window size (Pesaran and Timmermann, 2007, Journal of Econometrics)
- In other words, **how many observations should be used to estimate the parameter vector?**

Motivation

$$y_t = \alpha + \epsilon_t, \quad t = 1, 2, ..., 100$$



$$\hat{\mu}_1 = 9.90$$
$$\hat{\mu}_2 = 9.60, \qquad \tilde{\mu}_2 = 9.88$$
$$\hat{\mu}_3 = 7.20, \qquad \tilde{\mu}_3 = 9.40$$

Motivation

- Two most important papers on the optimal window selection
    - **Pesaran and Timmermann's (2007, Journal of Econometrics) cross-validation (PTCV) method**
        - selects the starting point of the window by partitioning data into two periods and comparing the recursive pseudo out-of-sample forecasts
        - requires strictly exogenous regressors and uncorrelated errors
        - suffers selection bias, when a break occurring shortly before the date of making forecasts distorts the ranking in the validation
    - **Inoue, Jin, and Rossi's (2017, Journal of Econometrics) algorithm (IJR)**
        - allows weak dependence and multi-step ahead forecasting
        - suffers selection bias, combining PTCV method

Contribution

- Propose two alternative algorithms developed from IJR's framework
  - Bootstrap Method
  - Simple Selection Method
- Keep the desired properties of the original method
  - Weak dependence
  - Multi-step ahead forecasting
  - Asymptotic validity

Model Framework

- Suppose we forecast y_{T+h} at time T
- The optimal forecast is given by

$$\hat{y}_{T+h} = x'_T \hat{\beta}_{\hat{R}} \tag{1}$$

- $\hat{\beta}_{\hat{R}}$ is the OLS estimates, using the most recent $\hat{R}$ observations (known as the window size)

Model Framework

- The optimal window size $\hat{R}$ is given by

$$\hat{R} \equiv \quad arg \min_{R \in \Theta_R} [\hat{\beta}_R(1) - \tilde{\beta}(1)]' x_T x_T' [\hat{\beta}_R(1) - \tilde{\beta}(1)] \qquad (2)$$

  where

$$\begin{bmatrix} \tilde{\beta}(1) \\ \tilde{\beta}^{(1)}(1) \end{bmatrix} = \begin{bmatrix} \sum x_t x_t' & \sum x_t x_t' (\frac{t-T}{T}) \\ \sum x_t x_t' (\frac{t-T}{T}) & \sum x_t x_t' (\frac{t-T}{T})^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum x_t y_{t+h} \\ \sum x_t y_{t+h} (\frac{t-T}{T}) \end{bmatrix} \qquad (3)$$

  - $\sum$ represents $\sum_{t=T-S+1}^{t=T-h}$
  - $S \geqslant 2k$ is an arbitrary number

- **The choice of S matters!**
- IJR chooses S using PTCV method
  - it may suffer from selection bias
  - its forecasting performance can be improved furthermore

Proposed Bootstrap Method

- Consider an optimization problem

$$S^{\star} \equiv arg \min_{S \in \Psi} \sum_{m=1}^{B} (y_{T+h}^{(m)} - \hat{y}_{T+h|T,S}^{(m)})^2 \quad (4)$$

  where
  - $y_{T+h}^{(m)}$ is the outcome at time $T+h$ for the m-th replication
  - $\hat{y}_{T+h|T,S}^{(m)}$ is the h-step ahead forecast at time T under S for the m-th replication
  - $\Psi = \{s\}_{s=2k}^{T}$ is the set of S
- B is the number of bootstrap re-sampling

Proposed Bootstrap Method

1. Partition the data into two periods according to the break date T_b as $\{y_t, x_t\}_{t=1}^{T_b}$ and $\{y_t, x_t\}_{t=T_b+1}^{T}$
2. Estimate parameter vectors $\hat{\beta}_1$ and $\hat{\beta}_2$ by OLS
3. Compute residuals $\{\hat{\epsilon}_{1,t}\}_{t=1+h}^{T_b}$ and $\{\hat{\epsilon}_{2,t}\}_{t=T_b+1+h}^{T}$
4. Centre estimated residuals as the empirical distribution function (EDF) E_1 and E_2
5. Resample residuals with replacement from the EDFs
    a. resample T_b residuals $\{\epsilon^{\star}_{1,t}\}_{t=1+h}^{T_b+h}$ from E_1
    b. resample $(T - T_b)$ residuals $\{\epsilon^{\star}_{2,t}\}_{t=T_b+1+h}^{T+h}$ from E_2

Proposed Bootstrap Method

6. Generate a bootstrap sample $\{y_t^\star\}_{t=1}^{T+h}$ with updates
   a. $y_{t+h}^\star = \hat{\beta}_1' x_t^\star + \epsilon_{1,t+h}^\star, \quad t = 1, 2, \cdots, T_b$
   b. $y_{t+h}^\star = \hat{\beta}_2' x_t^\star + \epsilon_{2,t+h}^\star, \quad t = T_b + 1, T_b + 2, \cdots, T$
7. Repeat steps 5-6, and generate B bootstrap samples, containing the information of the break in the original series
8. Apply (4) to choose the estimation window size for $\tilde{\beta}(1)$, S
9. Using S in step 8, apply (2) and (3) to select the optimal window size for forecasting

Proposed Simple Selection Method

- Concerning the computation burden of introducing the bootstrap, simplify the decision rule
- Estimate $\tilde{\beta}(1)$ using only post-break data
- $S = T - T_b$
- In practice, the break dates can be estimated by using the Sup-F test in Bai and Perron (1998, Econometrica)

Table: Comparison of Four Methods

Method	PTCV	IJR	Bootstrap	Simple Selection
Lagged Dependent Variables	No	Allowed	Allowed	Allowed
Correlated Error Terms	No	Allowed	NA	Allowed
Multi-step Ahead Forecasts	No	Allowed	Allowed	Allowed
Computation Burden	Medium	Heavy	Extremely Heavy	Medium

Monte-Carlo Study

- **Object**
  Test the forecasting performance of the proposed methods against that of existing methods under a structural break
- **Experiment Design**
  - Data Generating Process (DGP)

$$\begin{bmatrix} y_{t+1} \\ w_{t+1} \end{bmatrix} = \begin{bmatrix} a_t & b_t \\ 0 & 0.9 \end{bmatrix} \begin{bmatrix} y_t \\ w_t \end{bmatrix} + \begin{bmatrix} \mu_{t+1} \\ \upsilon_{t+1} \end{bmatrix} \tag{5}$$

where

$$\begin{bmatrix} \mu_{t+1} \\ \upsilon_{t+1} \end{bmatrix} \sim i.i.N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

  - A break on either a_t or b_t at time T_b is engaged
  - Various setups on break size and break date (T_b) are used

Monte-Carlo Study

- **Forecast Methods**
  - Post-break Method ("PB")
  - PT's CV Method ("PTCV")
  - IJR Method ("IJR")
  - Proposed Bootstap Method ("My1")
  - Proposed Simple Selection Method ("My2")

Results

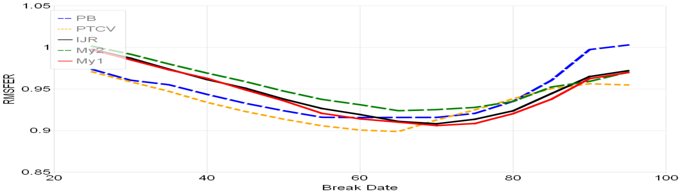
- Sample size $T = 100$
- One-step ahead forecasting practice $h = 1$
- 5000 Monte-Carlo simulations
- Benchmark: forecasts using the whole sample
- Criterion of forecast performance: ratio of square roots of MSFE (RMSFER)

$$\sqrt{\frac{\sum_{m=1}^{5000}(y_{T+1}^{(m)} - \hat{y}_{T+1}^{(m)})^2}{\sum_{m=1}^{5000}(y_{T+1}^{(m)} - \tilde{y}_{T+1}^{(m)})^2}}, \tag{6}$$

Results

- A small break on AR parameter with varying break date

Figure: RMSFER against break date

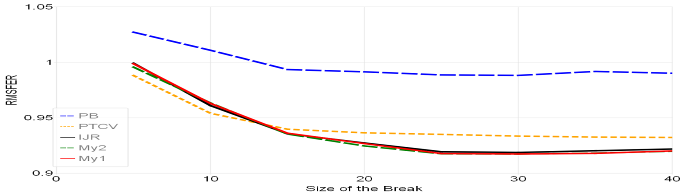


- "PTCV" dominates when the break date is before $0.65T$
- "My1" dominates when the break date is at $0.7T \sim 0.85T$
- "PTCV" dominates again when the break date is after $0.9T$

Results

- A break on AR parameter with varying break size at $T_b = 90$

Figure: RMSFER against break size

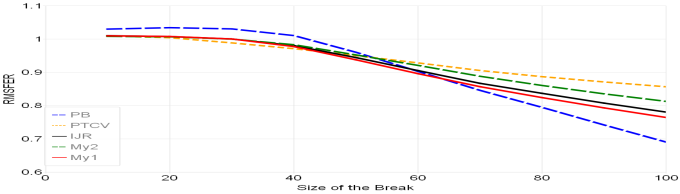


- Proposed "My1" and "My2" dominate others when the AR parameter shifts down by $0.15 \sim 0.4$

Results

- A break on marginal coefficient with varying break size at $T_b = 85$

Figure: RMSFER against break size



- "PTCV" dominates when the break size is small
- "My1" dominates when the break size is medium
- "PB" dominates when the break size is large

Conclusion

- The proposed bootstrap method outperforms IJR's original method in almost all cases
- The proposed bootstrap method performs best when there is a medium break close to the date of making forecasts
- If the break date is close to the forecast date, a small trimming value (e.g. 0.05) in Bai and Perron's (1998, Econometrica) test is preferred when using my bootstrap method.
- The proposed simple selection method performs well when the break occurs very close to the date of making forecasts
- When the break size is significant and the break date is far from the date of making forecasts, using post-break data only is almost always the best strategy

Discussion

- Caveats
  - What if there are more than one break (multiple breaks)
  - What if the parameter is time-varying
  - Extension to asymmetric loss function
  - When there exists weak dependence, the bootstrap may not be valid
    - Residual autocorrelation
    - Heteroscedasticity
  - Neither I or IJR investigated the ratio of the shift in mean and the variance

Thank you!

References I

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INOUE, A., L. JIN, AND B. ROSSI (2017): "Rolling window selection for out-of-sample forecasting with time-varying parameters," *Journal of Econometrics*, 196(1), 55–67.

PESARAN, M. H., AND A. TIMMERMANN (2007): "Selection of estimation window in the presence of breaks," *Journal of Econometrics*, 137(1), 134–161.